Steady-states and kinetics of ordering in bus-route models: connection with the Nagel-Schreckenberg model

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Abstract. A Bus Route Model (BRM) can be defined on a one-dimensional lattice, where buses are represented by "particles" that are driven forward from one site to the next with each site representing a bus stop. We replace the random sequential updating rules in an earlier BRM by parallel updating rules. In order to elucidate the connection between the BRM with parallel updating (BRMPU) and the Nagel-Schreckenberg (NaSch) model, we propose two alternative extensions of the NaSch model with space-/time-dependent hopping rates. Approximating the BRMPU as a generalization of the NaSch model, we calculate analytically the steady-state distribution of the time headways (TH) which are defined as the time intervals between the departures (or arrivals) of two successive particles (*i.e.*, buses) recorded by a detector placed at a fixed site (*i.e.*, bus stop) on the model route. We compare these TH distributions with the corresponding results of our computer simulations of the BRMPU, as well as with the data from the simulation of the two extended NaSch models. We also investigate interesting kinetic properties exhibited by the BRMPU during its time evolution from random initial states towards its steady-states.

PACS. 05.40.-a Fluctuation phenomena, random processes, noise, and Brownian motion – 05.60.-k Transport processes – 89.40.+k Transportation

1 Introduction

Systems of interacting particles driven far from equilibrium are of current interest in statistical physics [1–5]. Microscopic models of such systems often capture some aspects of vehicular traffic. In such "particle-hopping" models of vehicular traffic the particles represent vehicles and the nature of the interactions among these particles is determined by the manner in which the vehicles influence the motion of each other [6–8]. The dynamics of these models are often formulated in terms of "update rules" using the language of cellular automata (CA) [9]. For example, the Nagel-Schreckenberg (NaSch) model [10,11] is the most popular minimal CA model of vehicular traffic on highways while, to our knowledge, the first CA model of city traffic was developed by Biham, Middleton and Levin [12]. The results obtained for these models, using the techniques of statistical mechanics, are not only of fundamental interest for understanding truly nonequilibrium phenomena but may also find practical use in traffic science and engineering. [13–15]. Among such results are the time-headway and distance-headway distributions. The time-headway (TH) is defined as the time interval between the departures (or arrivals) of two successive

vehicles recorded by a detector placed at a fixed position on the route while the distance between the successive vehicles can be defined as the corresponding distance-headway (DH). The distributions of TH and DH not only contain detailed informations on the nature of the spatio-temporal organization of the vehicles but are also of practical interest to traffic engineers because larger headways provide greater margins of safety whereas higher capacities of the highway require smaller headways.

In a 1998 paper, O'Loan *et al.* [16] have developed a one-dimensional lattice model of bus-route where the buses are represented by particles which move from one site to the next; each site of this model represents a bus stop along the route. The motion of the buses in this bus route model with random sequential updating (BRMRSU) is strongly influenced by the passengers waiting at the bus stops. The BRMRSU model may be viewed as a generalization of a simple particle-hopping model, namely, the totally asymmetric simple exclusion process (TASEP) by coupling the dynamics of the particles to another new variable which represents the presence (or absence) of passengers waiting at the bus stops. The bus route model in [16] does not deal with overcrowded buses; it implicitly assumes that either the buses have infinite capacity or that the passenger arrival rate is slow enough to avoid overcrowding.

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The BRMRSU exhibits a Bose-Einstein-condensation-like phenomenon which has been observed earlier in the TASEP and in the NaSch model when quenched random hopping rates are associated with the particles [17–19]. However, unlike the stable Bose-Einstein-condensed states observed at sufficiently low densities in the TASEP (and in the NaSch model) with random hopping rates, those in the BRMRSU are metastable. The main characteristic of the spatially-inhomogeneous Bose-Einstein-condensed state is the existence of a macroscopically long gap in front of a cluster of vehicles led by the slowest one. In finite systems, for small λ (λ is the rate of passenger arrival at a bus stop), the bus clusters (or, equivalently, the gaps between clusters) in the BRMRSU exhibit interesting coarsening phenomena as the system evolves from a random initial state. O'Loan *et al.* [16] find that, after sufficiently long time, the typical size of the large gaps in the system grows with time t according to a power growth law $t^{1/2}$.

In this paper we use a BRM with parallel updating (BRMPU) which is obtained from the BRMRSU [16] by replacing the random sequential updating rule with parallel updating, with the aim of relating it with the NaSch model where updating is done in parallel. We also propose here two extensions of the NaSch model (from now onwards referred to as the models Y and Z) by replacing the constant hopping rates with two different time-/space-dependent hopping rates which we shall specify explicitly in Section 2. We have computed the TH distributions in the steady-states of the BRMPU as well as in the models Y and Z through computer simulations. Comparison of these distributions are shown in section III. Such comparisons elucidate the connection between the BRMPU and the NaSch model. Then, approximating the BRMPU as a generalization of the NaSch model with a time-dependent hopping rate for the buses, we calculate in Section 4 the TH distribution in the BRMPU from the corresponding analytical expression in the NaSch model. We compare the TH distributions thus derived from analytical considerations with the corresponding results of computer simulations of the BRMPU. These comparisons do not merely point out the regimes of validity of our analytical results but also indicate the differences arising from the different natures of the low-density steady-states in the BRMPU and the NaSch model. Finally, we investigate in Section 5, interesting kinetic phenomena at low densities of the BRMPU by computing the appropriate correlation functions (to be defined in Sect. 5). We extract the universal laws governing the growth of the clusters of buses in finite samples of BRMPU at low densities where the system approaches a Bose-Einstein-like "condensed" state evolving from random initial states.

2 The models and methods

Let us first summarize how the totally asymmetric exclusion process (TASEP) [1–3], the NaSch model [10] and the bus route models [16] are defined.

2.1 TASEP and the NaSch model

In the "particle-hopping" models of traffic the position, speed, acceleration as well as time are treated as discrete variables. In this approach, a lane is represented by a one-dimensional lattice. Each lattice site represents a "cell" which can be either empty or occupied by at most one "vehicle" at a given instant of time. At each discrete time step $t \rightarrow t + 1$, the state of the system is updated following a well-defined prescription. In the TASEP a randomly chosen particle can move forward, by one lattice spacing, with probability q if the lattice site immediately in front of it is empty. In the NaSch model, the speed v of each vehicle can take one of the $v_{\max} + 1$ allowed integer values $v = 0, 1, ..., v_{\max}$. If the random-sequential updating scheme of the TASEP is replaced by parallel updating then it becomes identical to the NaSch model with $v_{\max} = 1$ and random braking probability $p = 1 - q$. Our interest in the NaSch model is to unravel its connections to the BRM. For this purpose, we only need the NaSch model with $v_{\max} = 1$. Thus in what follows, by the NaSch model, we shall mean NaSch model with $v_{\max} = 1$, unless explicitly stated otherwise.

2.2 BRM with parallel and random-sequential updatings

In the BRM [16] each of the lattice sites represents a bus stop and these stops are labeled by an index i ($i = 1, 2, ..., L$) [16]. In each step of updating, each bus attempts to hop from one stop to the next. Note that in the TASEP and the NaSch model one can label the lattice sites by the index i ($i = 1, 2, ..., L$) and describe the state of each of the sites by associating a variable σ_i with it; $\sigma_i = 1$ if the site i is occupied and $\sigma_i = 0$ if the site i is empty. In contrast, in the BRM, two binary variables σ_i and ϕ_i are assigned to each site i: (i) If the site i is occupied by a bus then $\sigma_i = 1$; otherwise $\sigma_i = 0$. (ii) If site i has passengers waiting for a bus then $\phi_i = 1$; otherwise $\phi_i = 0$. A site cannot have both $\sigma_i = 1$ and $\phi_i = 1$ simultaneously since a site cannot have simultaneously a bus and waiting passengers. The state of the system is updated according to the following random sequential update rules: a site i is picked up at random. Then, (i) if $\sigma_i = 0$ and $\phi_i = 0$ (*i.e*, site i contains neither a bus nor waiting passengers), then $\phi \rightarrow 1$ with probability λ, where λ is the probability per unit time of the arrival (*i.e.* the arrival rate) of the first passenger at the empty bus stop. (Arrival of the subsequent passengers does not affect the time evolution.) (ii) If $\sigma_i = 1$ (*i.e.*, there is a bus at the site i) and $\sigma_{i+1} = 0$, then the hopping rate μ of the bus from site i to $i + 1$ is defined as follows: (a) if $\phi_{i+1} = 0$, then $\mu = \alpha$ but (b) if $\phi_{i+1} = 1$, then $\mu = \beta$, where α is the hopping rate of a bus onto a stop which has no waiting passengers and β is the hopping rate onto a stop with waiting passenger(s). Generally, $\beta < \alpha$, which reflects the fact that a bus has to slow down when it has to pick up passengers. In the BRMRSU one can set $\alpha = 1$ without loss of generality. However, for reasons which will become clear soon, we shall keep $\beta < \alpha < 1$. When a bus

hops onto a stop i with waiting passengers, ϕ_i is reset to zero as the bus takes all the passengers. Note that the density of buses $c = N/L$ is a conserved quantity whereas that of the passengers is not.

In the BRMPU the random sequential update rules of the BRMRSU are replaced by parallel updating but all the other aspects of the updating remain unchanged. BRMPU is related to the NaSch model in two extreme limits of λ: In the unphysical limit of $\lambda = 0$ (which means the passengers never arrive at a bus stop), the BRMPU reduces to the NaSch model with $v_{\max} = 1$ and $q = 1 - p = \alpha$. In the opposite limit of maximum value of λ in BRMPU, $\lambda = \infty$ (very fast rate of passenger arrival at a bus stop), the BRMPU is equivalent to the NaSch model with $v_{\max} = 1$ and $q = 1 - p = \beta$. Note that since the time between two updating steps is the unit of time, all values of $\lambda \geq 1$ are synonymous with $\lambda = \infty$. This is because any value of $\lambda \geq 1$ will bring at least one passenger to an empty bus stop between two updating times. Interesting results in the model occur only for values of $\lambda \ll 1$.

Note that if we take $\alpha = 1$, then the limit $\lambda = 0$ would correspond to the limit $q = 1$ (*i.e.*, $p = 0$) of the NaSch model which is a deterministic CA and does not exhibit jammed states [20]. Since we are interested in exploring the connection between the BRMPU and the NaSch model with arbitrary q throughout this paper we consider $\alpha < 1$.

2.3 Extended NaSch models

It has been realized over the last few years that different modifications of the braking rule in the NaSch model can lead to different types of phenomena which are interesting from the perspective of statistical physics. For example, such modifications can lead to self-organized criticality [21] as well as metastability and phase segregation [22,23]. Klauck and Schadschneider [24] considered a model where the particle is allowed to hop forward by one site or by two sites with two different hopping rates. It has also been established that assigning quenched random hopping rates can lead to the formation of clusters of vehicles [17–19].

In a similar vein, we now extend the NaSch model by replacing its constant (time-independent) hopping rate q by two other alternatives which are intended to mimic the situations in the BRMPU.

In one of these two alternatives (from now onwards referred to as model Y) the hopping rate of a vehicle at a given site x is given by

$$q_x = \beta + (\alpha - \beta)e^{-\Lambda T_{x+1}} \qquad (1)$$

where $\Lambda > 0$ is a constant and T_{x+1} is the time interval that has elapsed since the leading vehicle (LV) left the site $x + 1$. T_{x+1} is therefore the time interval between the departure of LV and the arrival of the following vehicle (FV) at the site $x + 1$.

In the other extended NaSch model (from now onwards referred to as model Z) the hopping rate of the n-th vehicle depends on its instantaneous DH Δx_n:

$$q_n = \beta + (\alpha - \beta)e^{-\Lambda\,\Delta x_n/\beta} \qquad (2)$$

where β (< 1) is a constant. At first sight it may seem more appropriate to have v, rather than β, in the exponential in equation (2). However, in Section 3 and Figure 3b, we show that the extended NaSch model with the hopping rates of the form (2) is a good approximation of the BRMPU in a wide range of circumstances.

Both the models Y and Z reduce to the NaSch model with constant hopping rates α in the limit $\Lambda = 0$, and reduce to the NaSch model with a constant hopping rate β in the limit $\Lambda = \infty$.

Models Y and Z are devised to capture the essential features of a bus-route model where the time- /space-dependent hopping rates of the vehicles depend on the presence or the absence of waiting passengers.

2.4 Methods of simulation

For the numerical calculations of the various quantities through computer simulations, we let the system evolve from a random initial state following the appropriate updating rules mentioned above. We compute the quantities relevant for the investigation of the kinetics of the system during the time-evolution of the system towards its steady-state. After the system reaches steady-state, we compute its steady-state properties, *e.g.*, the TH distribution, by letting it evolve for the next 5×10^4 time steps to obtain the required data. We then repeat the calculation with a different random initial state and, finally, average the data over 100 different random initial states of the system.

The largest systems we have simulated have a total length $L = 10^5$; each sample of these was allowed to evolve up to a maximum of 10^6 time steps which is not long enough to reach the corresponding steady-state but were used for the study of the kinetics. For the computation of the average steady-state properties we have used smaller systems (typically $L = 10^4$) which require shorter time to reach steady-state. In all our simulations we have used a periodic boundary condition.

3 Results of the extended NaSch models Y and Z

In Figure 1 we plot the TH distributions in the models Y and Z for $\Lambda = 0.01$, $\alpha = 0.9$, $\beta = 0.5$ at two different densities, namely, $c = 0.1$ and $c = 0.5$. Note that in the special case $\Lambda = 0$, both the models Y and Z as well as the BRMPU reduce to the NaSch model with $q = \alpha$. The data in Figure 1 establish that, when Λ is sufficiently small (*e.g.*, $\Lambda = 0.01$), the results of the models Y and Z agree well with those of the BRMPU at all densities for identical values of the set of parameters.

In order to emphasize the effects of time-/space-dependence of the hopping rates on the TH distribution

(a)   (b)
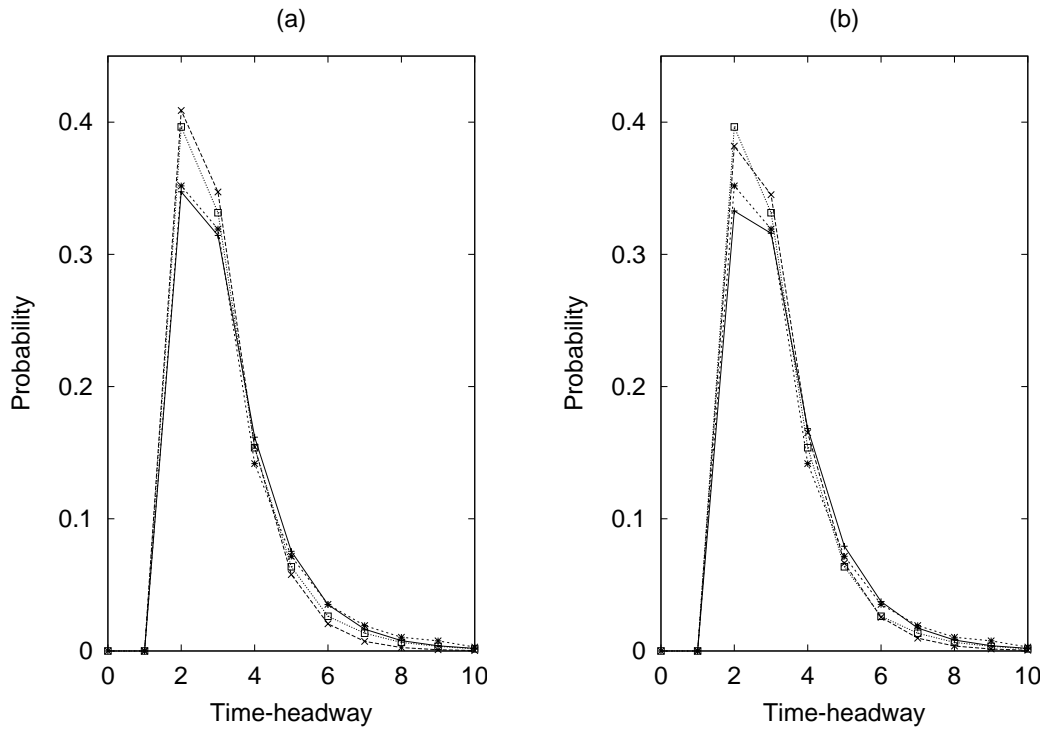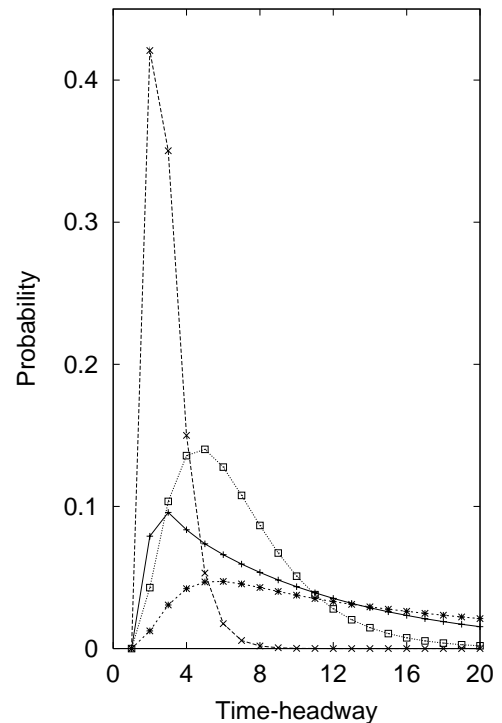

Fig. 1. The TH distributions in the (a) model Y, and (b) model Z for densities $c = 0.1(+)$ and $c = 0.5(\times)$ are compared against the corresponding distributions in the BRMPU for $c = 0.1(*)$ and $c = 0.5(\square)$. The common parameters are $\Lambda = \lambda = 0.01$, $\alpha = 0.9$ and $\beta = 0.5$. The discrete symbols denote the numerical data obtained through computer simulations while the continuous lines joining these points serve merely as guides to the eye.

we plot in Figure 2 the exact TH distributions in the NaSch model with $v_{\max} = 1$ for $q = 0.9$ and $q = 0.5$ at the same two densities as used in Figure 1. The observation that the TH distribution in the NaSch model for $q = 0.9, c = 0.5$ is narrow can be explained by fact that small noise $(p = 0.1)$ gives rise to only a small width of the δ-function-like TH distribution, centered at 2, that one would observe at $c = 0.5$ in the deterministic limit $q = 1.0$ of the NaSch model. Comparing Figure 2 with the Figure 1 we find that, except for $q = 0.9, c = 0.1$, the TH distributions in the NaSch model have much longer tail than those in the BRMPU as well as in the model Y and model Z for the parameters $\alpha = 0.9, \beta = 0.5$.

Thus, the BRMPU is well-approximated by both the models Y and Z at Λ as small as 0.01. However, we find a larger difference between the TH distributions in the models Y and Z at larger values of Λ (see Fig. 3a). For $\lambda \geq 1$ the BRMPU reduces to the NaSch model with $q = \beta$ and the corresponding TH distribution is in excellent agreement with that in the model Z but differs significantly from that in the model Y (see Fig. 3b).

The results in Figures 1, 2 and 3 show that in a certain density regime, the BRMPU is well approximated by the NaSch model with time-/space-dependent hopping rates. We expect the passenger arrival rate λ of the BRMPU and the hopping rate Λ in models Y and Z to be related. We assume that the two are equal, even though we continue to use the two different symbols in order to allow the possibility of a difference between them in future simulations.

Fig. 2. The TH distributions in the NaSch model for $q = 0.9, c = 0.5(\times)$; $q = 0.9, c = 0.1(+)$; $q = 0.5, c = 0.5(\square)$ and $q = 0.5, c = 0.1(*)$. The discrete symbols denote the exact results obtained analytically while the continuous lines joining these points serve merely as guides to the eye.
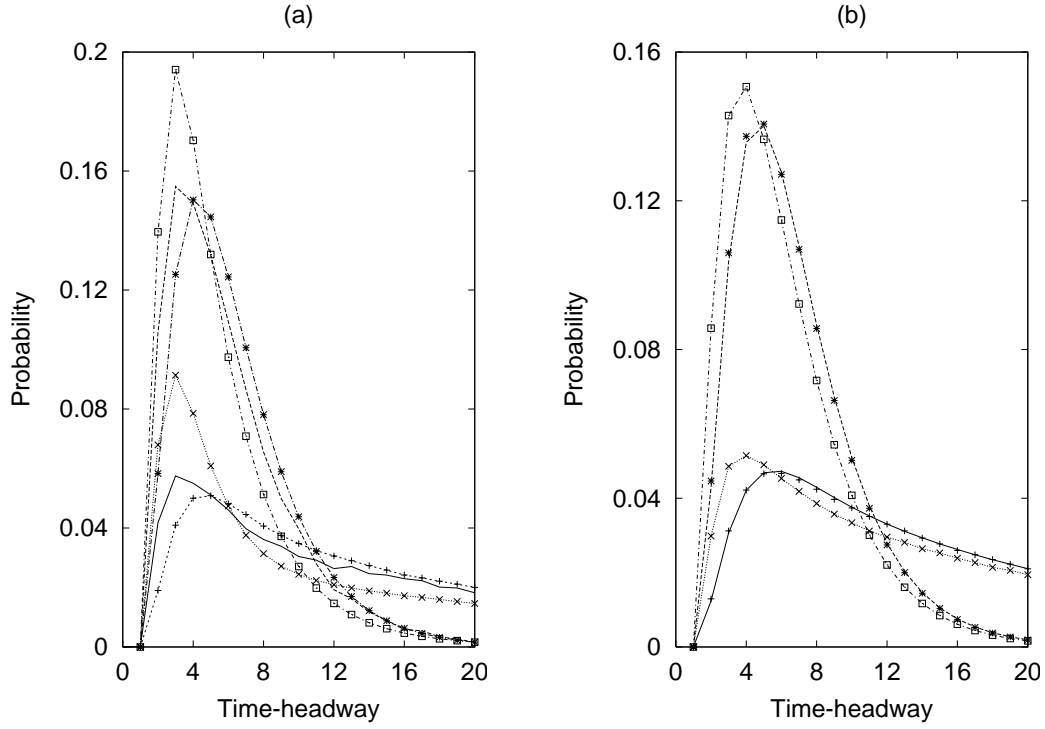
Fig. 3. The TH distributions in BRMPU, model Y and model Z for (a) $\Lambda = \lambda = 0.5$ and (b) $\Lambda = \lambda = 1.0$; the common parameters being $\alpha = 0.9, \beta = 0.5$. The continuous line and the dotted line correspond to $c = 0.1$ and $c = 0.5$, respectively, in BRMPU, while the discrete data points denoted by the symbols $+$ and $*$ correspond to $c = 0.1$ and $c = 0.5$, respectively, in the model Z. The symbols $\times$ (connected by dashed line) and $\square$ (connected by dashed-dotted line) correspond to $c = 0.1$ and $c = 0.5$, respectively, in the model Y.

4 Analytical results of the BRMPU

For analytical calculation of the TH distributions we label the site (*i.e.*, the bus stop) where the detector is located by $j = 0$, the stop immediately in front of it by $j = 1$, and so on. The detector clock resets to $\tau = 0$ every time a bus leaves the detector site. We begin our analytical calculations by writing $\mathcal{P}_{\text{th}}(\tau)$, the probability of a TH τ between the LV and the FV of a pair, as

$$\mathcal{P}_{\text{th}}(\tau) = \sum_{t_1=1}^{\tau-1} P(t_1)Q(\tau - t_1|t_1) \tag{3}$$

where $P(t_1)$ is the probability that there is a time interval t_1 between the departure of the LV and the arrival of the FV at the detector site and $Q(\tau - t_1|t_1)$ is the conditional probability given that the FV arrives at the detector site t_1 time steps after the departure of the LV, it halts for $\tau - t_1$ time steps at that site.

Encouraged by the success of the models Y and Z in capturing the TH distributions over moderate and high density regimes, we now approximate the BRMPU as an extended NaSch model with a time-dependent hopping rate which is closely related to (but slightly different from) those in the models Y and Z. Thus for our analytical calculation of $\mathcal{P}_{\text{th}}(\tau)$ in the BRMPU, we approximately treat it as an extended NaSch model (with $v_{\max} = 1$) where the hopping rate q, instead of being a constant, is a time-

dependent quantity given by the expression

$$q = \beta + (\alpha - \beta)e^{-\Lambda t_1}. \tag{4}$$

This form can be compared to those given in equations (1) and (2).

The exact analytical expression for $P(t_1)$ in the NaSch model (with $v_{\max} = 1$) has been derived earlier [25,26] using a 2-cluster approximation [11] which goes beyond the simple mean field approximation. Following the same arguments we now get

$$P_{\text{cl}}(t_1) = \mathcal{C}(1|\underline{0})q\left[\mathcal{C}(0|\underline{0})q + p\right]^{t_1-1} \tag{5}$$

where q is given by (4) and $\mathcal{C}$ gives the 2-cluster steady-state configurational probability for the argument configuration; the underline under an argument of $\mathcal{C}$ implies the associated condition. The expressions for the various $\mathcal{C}$s are given by [11,25,26]

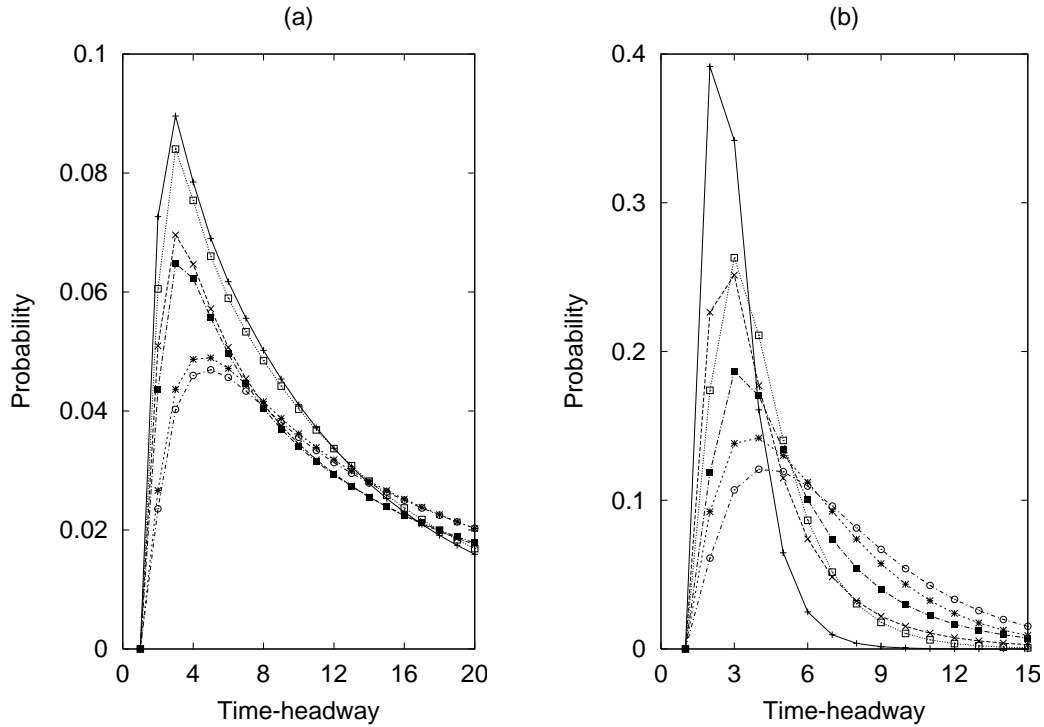$$\mathcal{C}(\underline{1}|0) = \mathcal{C}(0|\underline{1}) = \frac{y}{c} \tag{6}$$

$$\mathcal{C}(\underline{0}|1) = \mathcal{C}(1|\underline{0}) = \frac{y}{d} \tag{7}$$

$$\mathcal{C}(\underline{1}|1) = \mathcal{C}(1|\underline{1}) = 1 - \frac{y}{c} \tag{8}$$

$$\mathcal{C}(\underline{0}|0) = \mathcal{C}(0|\underline{0}) = 1 - \frac{y}{d} \tag{9}$$

where

$$y = \frac{1}{2q}\left(1 - \sqrt{1 - 4qcd}\right), \tag{10}$$

Fig. 4. The TH distributions in the BRMPU calculated analytically at densities (a) $c = 0.1$ (b) $c = 0.5$. The symbols $+, \times, (*)$ correspond to $\Lambda = 0.01, 0.10, 0.50$, respectively, in the 2-cluster approximation, while the symbols $\square, \blacksquare, \circ$ correspond to $\Lambda = 0.01, 0.10, 0.50$, respectively, in the simple mean field approximation. The common parameters are $\alpha = 0.9$ and $\beta = 0.5$. The discrete symbols denote the results obtained through analytic equation (3), while the continuous lines joining these points serve merely as guides to the eye. Note also the different scales along both the axes in (a) and (b).

$q = 1 - p$ and $d = 1 - c$.

On the other hand, in the simple mean field approximation, the 2-cluster probabilities reduce to $\mathcal{C}(1|\underline{0}) \rightarrow c$ and $\mathcal{C}(0|\underline{0}) \rightarrow 1 - c$ and, hence,

$$P_{\mathrm{mf}}(t_1) = cq[1 - cq]^{t_1 - 1}. \qquad (11)$$

We shall calculate the TH distribution, $\mathcal{P}_{\mathrm{th}}(\tau)$, given in (3), using the expression (5) (together with (7), (9) and (10)) and then compare with the corresponding TH distribution obtained by using (11), instead of (5), to emphasize the importance of correlations.

In order to obtain $\mathcal{P}_{\mathrm{th}}(\tau)$, let us next calculate $Q(\tau - t_1 | t_1)$. Again, following the arguments used earlier [26] in the calculation of the TH distribution in the NaSch model, we get

$$Q(\tau - t_1 | t_1) = (1 - \bar{g}^{t_1})p^{\tau - t_1 - 1}q$$
$$+ \bar{g}^{t_1}gq\frac{[(\bar{g})^{\tau - t_1 - 1} - (p)^{\tau - t_1 - 1}]}{\bar{g} - p} \qquad (12)$$

where g is the probability that a vehicle moves in the next time step (*i.e.*, in the $(t + 1)^{\mathrm{th}}$ time step) and $\bar{g} = 1 - g$. In the 2-cluster approximation

$$g_{\mathrm{cl}} = q\mathcal{C}(\underline{1}|0) \qquad (13)$$

which, in the simple mean field approximation reduces to

$$g_{\mathrm{mf}} = q(1 - c). \qquad (14)$$

Substituting (5) and (12) into (3) and using (13) for g and (4) for q (together with (6–9) for the configurational probabilities and (10)), we get $\mathcal{P}_{\mathrm{th}}(\tau)$ in the 2-cluster approximation by carrying out the summation over t_1 in (3) numerically. We shall refer to this result as the 2-cluster estimate of the TH distribution. Similarly, substituting (11) and (12) into (3) and using (14) for g and (4) for q (together with (6–9) and (10)) we get the simple mean field estimate of $\mathcal{P}_{\mathrm{th}}(\tau)$ by again summing over t_1 numerically. Note that in both cases, $\mathcal{P}_{\mathrm{th}}(\tau)$, in addition to its τ dependence, depends on parameters $c, \alpha, \beta, \Lambda$.

In Figure 4 we show these analytic results for three different values of Λ and two different values of densities c. At sufficiently low density of buses, there is hardly any difference between the 2-cluster estimate and the simple mean field estimate of the TH distribution (see Fig. 4a). However, with the increase of the density of the buses, the difference between these two estimates increases (see Fig. 4b).

As noted earlier, the TH distribution in the BRMPU changes continuously with the variation of λ; the results for $\lambda = 0$ and $\lambda = 1$ are identical to those in the NaSch model with $q = \alpha$ and $q = \beta$, respectively.

In Figure 5 we compare the 2-cluster analytic estimate of the TH distribution in the BRMPU (approximated as the extended NaSch model) for three different values of Λ, namely, $\Lambda = 0.01, 0.10, 0.50$, with the corresponding numerical data we have obtained from direct computer
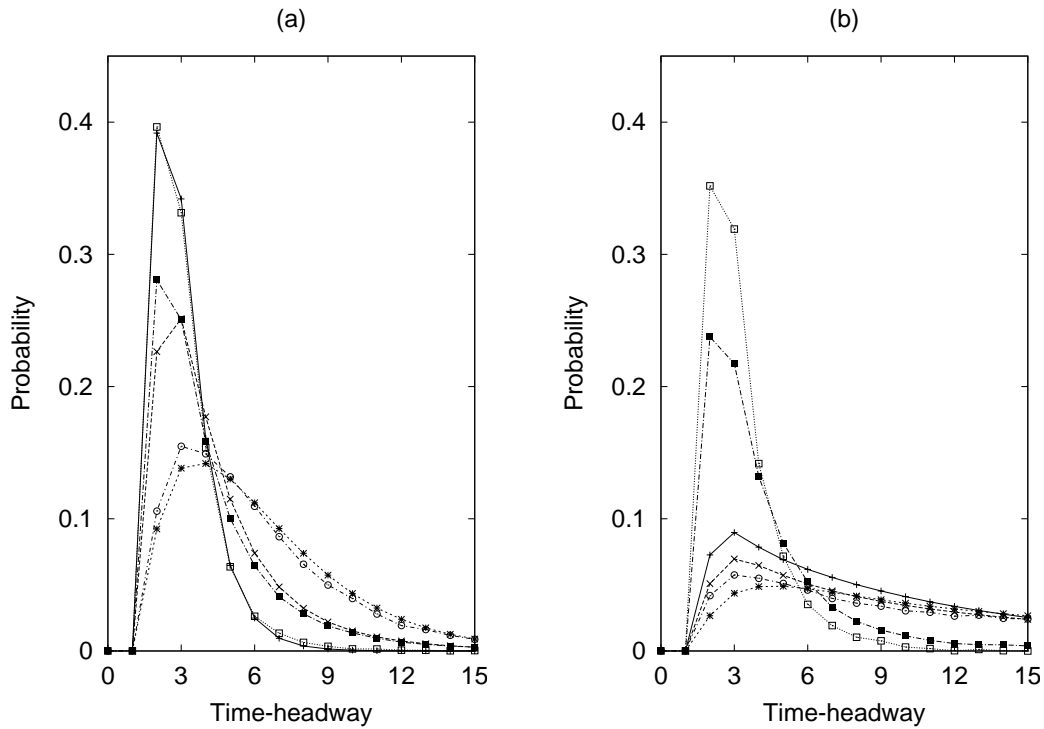
Fig. 5. The TH distributions in the BRMPU calculated analytically at densities (a) $c = 0.5$ (b) $c = 0.1$. The symbols $+, \times, (*)$ correspond to the TH distributions in the BRMPU, obtained analytically, for $\Lambda = 0.01, 0.10, 0.50$, respectively, in the 2-cluster approximation while the symbols $\square, \blacksquare, \circ$ correspond to the TH distributions in the BRMPU, obtained through computer simulations, for $\lambda = 0.01, 0.10, 0.50$, respectively. The common parameters are $\alpha = 0.9$ and $\beta = 0.5$. The continuous lines joining these points serve merely as guides to the eye. Note also the different scales along both the axes in (a) and (b).

simulations of the BRMPU model. Figure 5a shows the comparison for the higher density value $c = 0.5$. We note very good agreement between the 2-cluster estimates and the computer simulation data. Similar comparison at a lower density $c = 0.1$ is shown in Figure 5b. The poor agreement between the 2-cluster estimate and simulation data for the TH distribution in the BRMPU at low densities is a consequence of the fact that at low densities, the vehicles in a finite system form a cluster in the steady-state where there is at most one or two empty sites in between each pair of vehicles. It is well-known that the 2-cluster approximation scheme is not good enough for such states where correlation extends over distances which are much longer than what can be captured by a 2-cluster approximation. On the other hand, at higher densities there is no clustering of the buses in the steady-state of the BRMPU and the physics of the system is very similar to that in the steady-states of the NaSch model. This is because most of the buses stop on account of another bus at the next site, rather than due to waiting passengers.

Nevertheless, since the states with clusters of buses are metastable in infinitely long samples of BRMPU, the 2-cluster approximation is expected to yield good estimates for the stable steady-states of the BRMPU even at low densities. However, since it is extremely difficult (requires very long simulation time) to achieve these stable steady-states in any computer simulation at small λ we have not been able to demonstrate this explicitly.

5 Kinetics in the BRMPU

In traffic models like BRMPU, the kinetics are governed by two coupled fields, local passenger density $\phi_i(t)$ and local bus density $\sigma_i(t)$. In this paper a binary approximation (zero or nonzero) to the $\phi_i(t)$ field is made. $\sigma_i(t)$ is globally conserved and $\phi_i(t)$ is a nonconserved field. It is not clear whether the kinetics seen in our simulations of BRMPU and models Y and Z are derivable from a free energy functional. We are currently inquiring into such a possibility. If this turns out to be the case, then the model of kinetics appropriate to our simulations is model C [27] in the Halperin-Hohenberg classification scheme of critical dynamics.

Let us define the correlation function

$$\mathcal{C}(r,t) = \left[\frac{1}{L} \sum_{i=1}^{L} \sigma_i(t)\sigma_{i+r}(t) - c^2 \right] \qquad (15)$$

where $t = 0$ corresponds to the initial state. The symbol $[.]$ indicates average over random initial conditions. By definition, $\mathcal{C}(r,t)$ vanishes in the absence of any correlation in the occupation of the sites by the buses. Also at any time t, $\mathcal{C}(r = 0, t) = c(1-c)$. This correlation function has been calculated earlier for the NaSch model analytically for $v_{\max} = 1$ [28] and numerically for higher values of $v_{\max}$. However, the nature of this correlation function
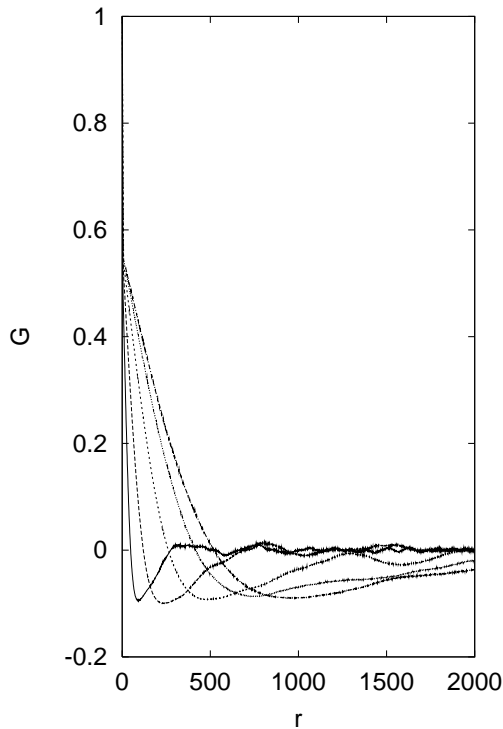
Fig. 6. The normalized correlation function $G(r,t)$ in the BRMPU plotted against r for (from left to right) $t = 5 \times 10^3$ (solid line), $t = 10^5$ (long dashes), $t = 10^6$ (short dashes), $t = 3 \times 10^6$ (dots), $t = 5 \times 10^6$ (dashed-dotted line) at a density $c = 0.1$. The values of the other parameters are $\lambda = 0.01, \alpha = 0.9, \beta = 0.5$.

Fig. 7. The log-log plot of $R(t)$ in the BRMPU at $c = 0.1$. The values of the other parameters are $\lambda = 0.01, \alpha = 0.9, \beta = 0.5$. The dashed (top) and dotted (bottom) lines have slopes of $1/2$ and $1/3$ respectively.

in the BRMPU is expected to differ qualitatively, particularly at low densities, from that in the NaSch model because of the formation of clusters in the steady-state of the BRMPU.
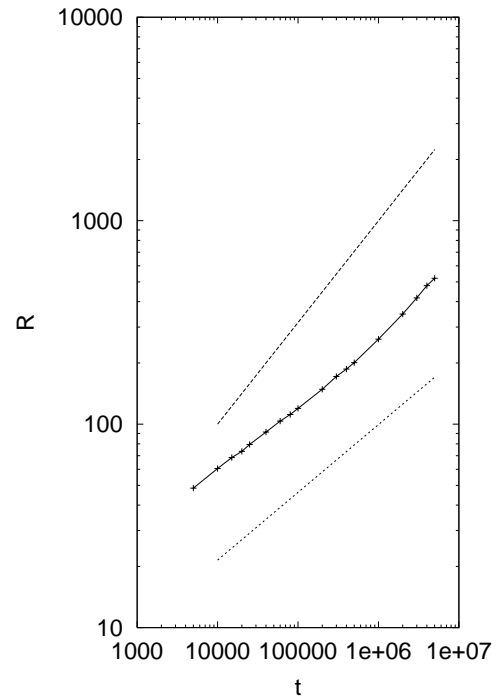
We compute $\mathcal{C}(r,t)$ during the time-evolution of the system from random initial states. Our simulations for the time evolution are done only for one set of parameters: $c = 0.1, \lambda = 0.01, \alpha = 0.9$ and $\beta = 0.5$. In Figure 6 we plot the normalized correlation function

$$G(r,t) = \mathcal{C}(r,t)/\mathcal{C}(r = 0,t) = \mathcal{C}(r,t)/(c(1-c)) \qquad (16)$$

as a function of r for values of t up to 5×10^6.

The value $r = R$ corresponding to the first zero-crossing of $G(r,t)$ is taken as a measure of the typical size of the clusters of buses at time t [29]. The fact that R increases with t indicates the coarsening of these clusters. It is worth mentioning here that in systems with conserved order parameters (the so-called model B, of which the binary alloy is a physical realization) the coarsening follows the Lifshitz-Slyozov law $R(t) \sim t^{1/3}$. In the case of the BRMPU, $R(t)$ may appear to follow the same Lifshitz-Slyozov law if the data up to $t \simeq 10^6$ are shown on a log-log plot (see Fig. 7). However, the upward turn of the data beyond $t \simeq 10^6$ in Figure 7 indicates more subtle features of this growth. In fact, fitting the raw data $R(t)$ to the curve

$$R(t) = R_0 + At^{1/2} \qquad (17)$$

we have estimated the parameters R_0 and A. We found that $R_0 \simeq 55$ and $A \simeq 0.2$. If the growth of $R(t)$, indeed, follows the law (17) then the $t^{1/2}$ growth law is expected to become clearly visible directly in Figure 7 for times long enough to satisfy the condition $At^{1/2} >> R_0$, i.e., for $t \gg 10^5$. This argument, together with our estimate $R_0 \simeq 55$, explains why the true growth law (17) can be anticipated in Figure 7 only beyond $t \simeq 10^6$. In fact, plotting $R(t)$ against $t^{1/2}$ and comparing with $55 + 0.2t^{1/2}$ in Figure 8 we do, indeed, see clear evidence of the $t^{1/2}$ growth for $t \gg 10^5$.

Finally, in Figure 9 we plot the normalized correlation function G against the scaled variable $r/R(t)$. Since the data for t as widely separated as $t = 5 \times 10^3$ to $t = 5 \times 10^6$ superpose, the validity of dynamic scaling [29] is convincing for kinetics of the BRMPU model. The $t^{1/2}$ growth in our BRMPU model that is akin to model C [27] implies that it is the nonconserved passenger density field ϕ that is driving the kinetics, for the parameter set that we have simulated.

6 Conclusions

The models and simulations presented in this paper were inspired by the work on BRMRSU in reference [16]. The model in [16] uses random sequential updating (RSU),
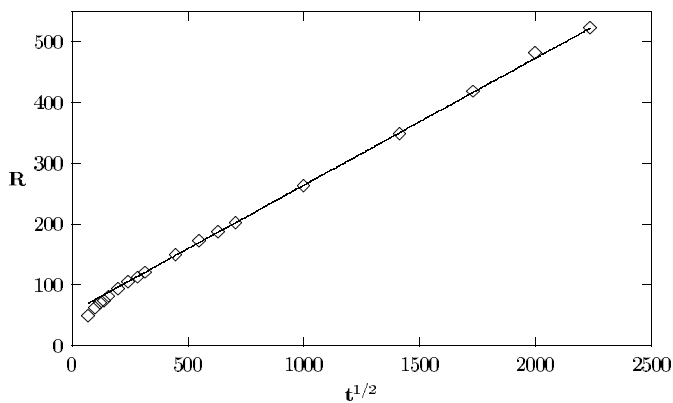
Fig. 8. The discrete data points represent $R(t)$ in the BRMPU at $c = 0.1$ plotted against $t^{1/2}$. The continuous solid line corresponds to $55 + 0.2t^{1/2}$. The values of the other parameters are $\lambda = 0.01, \alpha = 0.9, \beta = 0.5$.

whereas our complimentary study is based on parallel updating (PU). As expected, the properties of the steady states including the dependence of average velocity and current on the particle density are similar (not explicitly displayed in figures). We have in addition computed and analytically obtained (in the 2-cluster approximation) the time headway (TH) distributions for a wide range of parameters. At moderate and high densities, the simulation and analytic results agree well; but the comparison fails at low densities. We have also studied kinetics of the BRMPU which is also complimentary to that done for the BRMRSU in reference [16]. We find the bus clusters to grow in size as $t^{1/2}$. This shows that the growth exponent is robust with respect to the updating schemes. We have also computed the equal time pair correlation function of the local bus density, and find that it obeys a dynamical scaling ansatz: $G(r, t) = G(r/R(t))$, where $R(t)$ is the bus cluster size.

We have also found connection between the models of BRMPU type and the NaSch model. The NaSch model is the minimal CA model of vehicular traffic on idealized single-lane highways. This model has been extended [21–24,30–32] in various ways to incorporate some aspects of real traffic which are not captured by the minimal model. All the bus route models which we have considered in this paper, namely, the BRMPU, model Y and model Z, may be regarded as extensions of the NaSch model with $v_{\max} = 1$. In each of these models the dynamics of the vehicles are coupled to another non-conserved field, namely, the passenger density field ϕ, resulting in space- and time-dependent hopping rates of the vehicles. However, there is a difference in the length and time scales in the bus route models and in the NaSch model. In the NaSch model the motion of the vehicles from one cell to the next is described on a time scale which is roughly the reaction time of each driver. In contrast, the lattice constant in the BRM is of the order of the distance between successive stops on the bus route. Thus, each
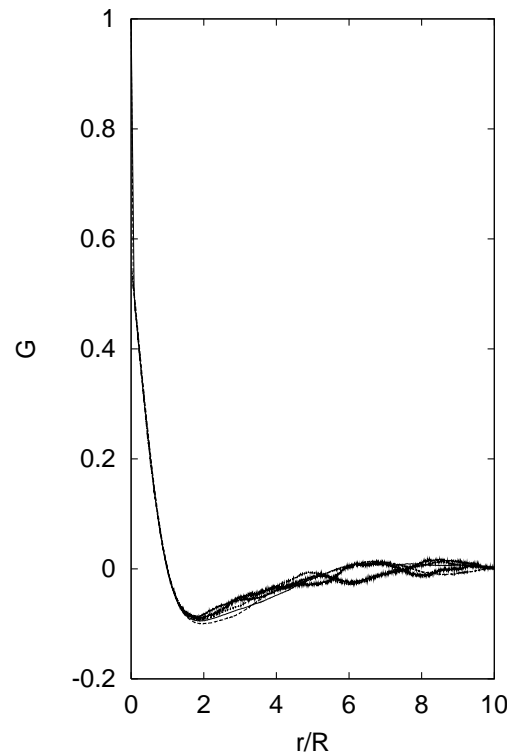
Fig. 9. The normalised correlation function G in the BRMPU at $c = 0.1$, for the same five values of t as in the Figure 6, are plotted against $r/R(t)$. The values of the parameters λ, α and β are also identical to those in the Figure 6.

time step corresponds to a much longer real time than that in the NaSch model. In other words, the interactions of the buses with the traffic, on its way from one stop to the next, are included in the BRM models only through the phenomenological rate constant α. It would be interesting to extend the bus route models further by including the interaction between a bus and other vehicles as it moves from one stop to the next. Inter-vehicle interactions are a natural ingredient of the NaSch model, and this prescription can be incorporated in the bus route models.

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